CURVATURE HOMOGENEOUS PSEUDO-RIEMANNIAN MANIFOLDS WHICH ARE NOT LOCALLY HOMOGENEOUS

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ABSTRACT. We construct a family of balanced signature pseudo-Riemannian manifolds, which arise as hypersurfaces in flat space, that are curvature homogeneous, that are modeled on a symmetric space, and that are not locally homogeneous.

1. INTRODUCTION

Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g) of signature (p, q). Following Kowalski, Tricerri, and Vanhecke [16, 17], we say that (M, g) is *curvature homogeneous* if given any two points $P, Q \in M$, there is a linear isomorphism $\Psi : T_P M \to T_Q M$ such that $\Psi^* g_Q = g_P$ and such that $\Psi^* R_Q = R_P$; this notion has also been called 0 curvature homogeneous when considering a similar condition for the higher covariant derivatives of the curvature tensor.

Similarly, (M, g) is said to be *locally homogeneous* if given any two points P and Q, there are neighborhoods U_P and U_Q of P and Q, respectively, and an isometry $\psi : U_P \to U_Q$ such that $\psi P = Q$. Taking $\Psi := \psi_*$ shows that locally homogeneous manifolds are curvature homogeneous. The somewhat surprising fact is that the converse fails – there are curvature homogeneous manifolds which are **not** locally homogeneous.

There is by now an extensive literature on the subject in the Riemannian setting, see, for example, the discussion in [1, 2, 14, 23, 24, 25]. There are also a number of papers in the Lorentzian setting [5, 6, 7] and also in the affine setting [15, 18]. There are, however, almost no papers in the higher dimensional setting – and those that exist appear in the study of 4 dimensional neutral signature Osserman manifolds, see, for example, [3, 8]. In this brief note, we exhibit a family of examples in signature (p, p) for any $p \geq 3$ which are curvature homogeneous but not locally homogeneous; this family first arose in the study of Szabó Osserman IP Pseudo-Riemannian manifolds [10, 11].

Let $(x, y) = (x_1, ..., x_p, y_1, ..., y_p)$ be the usual coordinates on $\mathbb{R}^{2p}$. Let $f(x)$ be a smooth function on an open subset $\mathcal{O} \subset \mathbb{R}^p$. We define a non-degenerate pseudo-Riemannian metric g_f of balanced signature (p, p) on $M := \mathcal{O} \times \mathbb{R}^p$ by:

(1.a) $\qquad g_f(\partial_i^x, \partial_j^x) = \partial_i^x f \cdot \partial_j^x f, \quad g_f(\partial_i^x, \partial_i^y) = \delta_{ij}, \quad \text{and} \quad g_f(\partial_i^y, \partial_j^y) = 0 \,.$

This is closely related to the so called 'deformed complete lift' of a metric on $\mathcal{O}$ to $T\mathcal{O}$, see, for example, the discussion in [4, 13, 20].

The pseudo-Riemannian manifold (M, g_f) arises as a hypersurface in a flat space. Let $\{\vec{u}_1, ..., \vec{u}_p, \vec{v}_1, ..., \vec{v}_p, \vec{w}_1\}$ be a basis for a vector space W. Define an inner product $\langle \cdot, \cdot \rangle$ of signature $(p, p + 1)$ on W by setting

$$\langle \vec{u}_i, \vec{u}_j \rangle = 0, \quad \langle \vec{u}_i, \vec{v}_j \rangle = \delta_{ij}, \quad \langle \vec{v}_i, \vec{v}_j \rangle = 0,$$
$$\langle \vec{u}_i, \vec{w}_1 \rangle = 0, \quad \langle \vec{v}_i, \vec{w}_1 \rangle = 0, \quad \langle \vec{w}_1, \vec{w}_1 \rangle = 1 \,.$$

Let $F(x, y) = x_1 \vec{u}_1 + ... + x_p \vec{u}_p + y_1 \vec{v}_1 + ... + y_p \vec{v}_p + f(x) \vec{w}_1$ define an embedding of M in W. Then g_f is the induced metric on the embedded hyper surface. The normal ν to the hypersurface is given by setting $\nu = \vec{w}_1 - \partial_1^x f \ \vec{v}_1 - ... - \partial_p^x f \ \vec{v}_p$. Thus the second fundamental form L_f of the embedding is given by the Hessian

$$L_f(\partial_i^x, \partial_j^x) = \partial_i^x \partial_j^x f, \quad L_f(\partial_i^x, \partial_j^y) = 0, \quad \text{and} \quad L_f(\partial_i^y, \partial_j^y) = 0 \,.$$

We define distributions

$$\mathcal{X} := \text{Span}\{\partial_1^x, ..., \partial_p^x\} \quad \text{and} \quad \mathcal{Y} := \text{Span}\{\partial_1^y, ..., \partial_p^y\}.$$

We then have $L(Z_1, Z_2) = 0$ if $Z_1 \in \mathcal{Y}$ or $Z_2 \in \mathcal{Y}$ so the restriction $L_f^{\mathcal{X}}$ of L to the distribution $\mathcal{X}$ carries the essential information. The following is the main result of this paper:

Theorem 1.1. *If the quadratic form $L_f^{\mathcal{X}}$ is positive definite, then (M, g_f) is curvature homogeneous. Furthermore, if $p \geq 3$, then (M, g_f) is not locally homogeneous for generic f.*

As noted above, these manifolds first arose in an entirely different setting. Let R be the Riemann curvature tensor of a pseudo-Riemannian manifold (M, g). Let ∇R be the covariant derivative of R. Let J, S, and $\mathcal{R}$ be the associated Jacobi operator, Szabó operator, and skew-symmetric curvature operator, respectively. Let $X \in TM$ and let $\{Y, Z\}$ be an oriented orthonormal basis for an oriented spacelike or timelike 2 plane π. These operators are defined by the identities:

$$g(J(X)U, V) = R(U, X, X, V),$$
$$g(S(X)U, V) = \nabla R(U, X, X, V; X),$$
$$g(\mathcal{R}(\pi)U, V) = R(Y, Z, U, V)$$

Stanilov and Videv [21] have defined a *higher order Jacobi operator* by setting

$$J(\pi) := g(X_1, X_1)J(X_1) + ... + g(X_\ell, X_\ell)J(X_\ell)$$

where $\{X_1, ..., X_\ell\}$ is any orthonormal basis for a non-degenerate subspace $\pi \subset TM$.

Definition 1.2. Let (N, g) be a pseudo-Riemannian manifold. Then (N, g) is

(1) *spacelike Jordan Osserman* (resp. *timelike Jordan Osserman*) if the Jordan normal form of $J(X)$ is constant on the bundle of unit spacelike (resp. unit timelike) vectors.

(2) *spacelike Szabó* (resp. *timelike Szabó*) if the eigenvalues of $S(X)$ are constant on the bundle of unit spacelike (resp. unit timelike) vectors.

(3) *spacelike Jordan IP* (resp. *timelike Jordan IP*) if the Jordan normal form of $\mathcal{R}(\pi)$ is constant on the Grassmannian of oriented spacelike (resp. timelike) 2 planes in TM.

(4) *Jordan Osserman* of type (r, s) if the Jordan normal form of $J(\pi)$ is constant on the Grassmannian of non-degenerate subspaces of type (r, s) in TM.

The spectral geometry of the Jacobi operator, of the skew-symmetric curvature operator, and of the Szabó operator were first considered in the Riemannian setting by Osserman [19], by Ivanova and Stanilov [12], and by Szabó [22], respectively. We refer to [9] for further details. The manifolds (M, g_f) provide examples of these manifolds. We refer to [10, 11] for the proof of the following result:

Theorem 1.3. *If the quadratic form $L_f^{\mathcal{X}}$ is positive definite, then (M, g_f) is spacelike Jordan Osserman, timelike Jordan Osserman, spacelike Szabó, timelike Szabó, spacelike Jordan IP, and timelike Jordan IP. Furthermore (M, g_f) is Jordan Osserman of types $(r, 0)$, $(0, r)$, $(p - r, p)$, and $(p, p - r)$, and is not Jordan Osserman of type (r, s) otherwise.*

We note there are no known Jordan Szabó manifolds which are not symmetric.

Here is a brief guide to the paper. In Section 2, we determine the tensors R_f and ∇R_f which are defined by the metric g_f and show (M, g_f) is curvature homogeneous. In Section 3, we complete the proof of Theorem 1.1 by showing that (M, g_f) is not locally homogeneous for generic f. We conclude in Remark 3.3 by showing the 'model space' for the curvature tensor for (M, g_f) is that of a symmetric space.

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2. The tensors R_f and ∇R_f

We begin the proof of Theorem 1.1 by determining R_f and ∇R_f.

Lemma 2.1. *Let $Z_1, ...$ be coordinate vector fields on $M := \mathcal{O} \times \mathbb{R}^p$. Let the metric g_f be given by Equation (1.a). Then:*
  (1) $\nabla_{Z_1} Z_2 = 0$ *if $Z_1 \in \mathcal{Y}$ or if $Z_2 \in \mathcal{Y}$;*
  (2) $R(Z_1, Z_2, Z_3, Z_4) = L(Z_1, Z_4)L(Z_2, Z_3) - L(Z_1, Z_3)L(Z_2, Z_4)$. *This vanishes if one of the $Z_i \in \mathcal{Y}$ for $1 \le i \le 4$;*
  (3) $\nabla R(Z_1, Z_2, Z_3, Z_4; Z_5) = Z_5\{R(Z_1, Z_2, Z_3, Z_4)\}$. *This vanishes if one of the $Z_i \in \mathcal{Y}$ for $1 \le i \le 5$.*

Proof. We have
$$(\nabla_{Z_1} Z_2, Z_3) = \tfrac{1}{2}\{Z_2 g_f(Z_1, Z_3) + Z_1 g_f(Z_2, Z_3) - Z_3 g_f(Z_1, Z_2)\}.$$
This vanishes if any of the $Z_i \in \mathcal{Y}$. Assertion (1) now follows. Let $g_{ij}^x := g(\partial_i^x, \partial_j^x)$ and let $\Gamma_{ijk}^x := \tfrac{1}{2}(\partial_i^x g_{jk}^x + \partial_j^x g_{ik}^x - \partial_k^x g_{ij}^x)$. We adopt the Einstein convention and sum over repeated indices to see
$$\nabla_{\partial_i^x}\partial_j^x = \Gamma_{ijk}^x \partial_k^y, \quad \nabla_{\partial_i^x}\partial_j^y = \nabla_{\partial_j^y}\partial_i^x = 0, \quad \text{and} \quad \nabla_{\partial_i^y}\partial_j^y = 0.$$
It now follows that $R(Z_1, Z_2, Z_3, Z_4) = 0$ if any of the $Z_i \in \mathcal{Y}$. Furthermore
$$R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = \partial_i \Gamma_{jkl}^x - \partial_j \Gamma_{ikl}^x.$$
Assertion (2) now follows; this also, of course, follows from the classical formula which expresses the curvature tensor of a hypersurface in flat space in terms of the second fundamental form.

Since $\nabla_{Z_5} Z_i \in \mathcal{Y}$ and since $R(\cdot, \cdot, \cdot, \cdot)$ vanishes if any of the entries belong to $\mathcal{Y}$, Assertion (3) follows from Assertion (2). $\qquad\square$

We show that (M, g_f) is curvature homogeneous by showing the following result:

Lemma 2.2. *Let $P \in M$. Assume $L_f^{\mathcal{X}}$ is positive definite. Then there exists a basis $\{X_1, ..., X_p, Y_1, ..., Y_p\}$ for $T_P M$ so that:*
  (1) $g_f(X_i, X_j) = 0$, $g_f(X_i, Y_j) = \delta_{ij}$, *and* $g_f(Y_i, Y_j) = 0$.
  (2) $R_f(X_i, X_j, X_k, X_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}$.
  (3) $R_f(\cdot, \cdot, \cdot, \cdot) = 0$ *if any of the entries is one of the vector fields $\{Y_1, ..., Y_p\}$.*

Proof. Fix $P \in M$. We diagonalize the quadratic form $L_f^{\mathcal{X}}$ at P to choose tangent vectors $\bar{X}_i = a_{ij}\partial_j^x \in T_P M$ so that $L(\bar{X}_i, \bar{X}_j) = \delta_{ij}$. Let $\bar{Y}_i := a^{ji}\partial_j^y$ where a^{ij} is the inverse matrix. Then
$$g_f(\bar{X}_i, \bar{Y}_j) = a_{ik}a^{\ell j}g_f(\partial_k^x, \partial_\ell^y) = a_{ik}a^{kj} = \delta_{ij},$$
$$g_f(\bar{Y}_i, \bar{Y}_j) = 0,$$
$$R_f(\bar{X}_i, \bar{X}_j, \bar{X}_k, \bar{X}_\ell) = \delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell},$$

and $R_f(\cdot,\cdot,\cdot,\cdot) = 0$ if any entry is $\bar{Y}_i$. We define

$$X_i := \bar{X}_i - \tfrac{1}{2} g_f(\bar{X}_i, \bar{X}_j) \bar{Y}_i \quad \text{and} \quad Y_i := \bar{Y}_i$$

to ensure $g_f(X_i, X_j) = 0$. It is immediate that the frame $\{X_1, ..., X_p, Y_1, ..., Y_p\}$ satisfies the normalizations of the Lemma. $\square$

3. Homogeneity

We begin our discussion with a technical observation. Let V be a finite dimensional real vector space. A 4 tensor $R \in \otimes^4 V^*$ is said to be an *algebraic curvature tensor* if it satisfies the symmetries of the Riemann curvature tensor, i.e. if:

$$R(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) = -R(\vec{v}_2, \vec{v}_1, \vec{v}_3, \vec{v}_4) = R(\vec{v}_3, \vec{v}_4, \vec{v}_1, \vec{v}_2) \quad \text{and}$$
$$R(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) + R(\vec{v}_2, \vec{v}_3, \vec{v}_1, \vec{v}_4) + R(\vec{v}_3, \vec{v}_1, \vec{v}_2, \vec{v}_4) = 0 \,.$$

If ϕ is a symmetric bilinear form on V, then we may define an algebraic curvature tensor R_ϕ on V by setting:

$$R_\phi(\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4) := \phi(\vec{v}_1, \vec{v}_4)\phi(\vec{v}_2, \vec{v}_3) - \phi(\vec{v}_1, \vec{v}_3)\phi(\vec{v}_2, \vec{v}_4) \,.$$

Lemma 3.1. *Let ϕ_1 and ϕ_2 be symmetric positive definite bilinear forms on a vector space V of dimension at least 3. If $R_{\phi_1} = R_{\phi_2}$, then $\phi_1 = \phi_2$.*

We note that Lemma 3.1 fails if $\dim V \le 2$.

Proof. Since ϕ_1 is positive definite, we can diagonalize ϕ_2 with respect to ϕ_1 and choose a basis $\{\vec{e}_1, ..., \vec{e}_r\}$ for V so that $\phi_1(\vec{e}_i, \vec{e}_j) = \delta_{ij}$ and so that $\phi_2(\vec{e}_i, \vec{e}_j) = \lambda_i \delta_{ij}$. If $i \ne j$, then

$$
\begin{aligned}
1 &= \phi_1(\vec{e}_i, \vec{e}_i)\phi_1(\vec{e}_j, \vec{e}_j) - \phi_1(\vec{e}_i, \vec{e}_j)\phi_1(\vec{e}_i, \vec{e}_j) = R_{\phi_1}(\vec{e}_i, \vec{e}_j, \vec{e}_j, \vec{e}_i) \\
\text{(3.a)} \qquad &= R_{\phi_2}(\vec{e}_i, \vec{e}_j, \vec{e}_j, \vec{e}_i) = \phi_2(\vec{e}_i, \vec{e}_i)\phi_2(\vec{e}_j, \vec{e}_j) - \phi_2(\vec{e}_i, \vec{e}_j)\phi_2(\vec{e}_i, \vec{e}_j) \\
&= \lambda_i \lambda_j \,.
\end{aligned}
$$

Since $r \ge 3$, we can choose k so $\{i, j, k\}$ are distinct indices. By Equation (3.a), $1 = \lambda_i \lambda_k = \lambda_j \lambda_k$ so $\lambda_i = \lambda_j$ for all i, j. Since $1 = \lambda_i \lambda_j = \lambda_i^2$ and since ϕ_2 is positive definite, $\lambda_i = 1$ for all i and hence $\phi_1 = \phi_2$. $\square$

We say that $\mathcal{B} := (X_1, ..., X_p, Y_1, ..., Y_p)$ is an *admissible* basis for $T_P M$ if $\mathcal{B}$ satisfies the normalizations of Lemma 2.2. We can now define a useful invariant:

Lemma 3.2. *Suppose that $L_f^{\mathcal{X}}$ is positive definite. Let $P \in M$ and let $\mathcal{B}$ be an admissible basis for $T_P M$. Let $\alpha_f(P, \mathcal{B}) := \sum_{i,j,k,l,n} \nabla R_f(X_i, X_j, X_k, X_l; X_n)(P)^2$.*

(1) $\alpha_f(P, \mathcal{B})$ is independent of the particular admissible basis $\mathcal{B}$ which is chosen.
(2) If (M, g_f) is locally homogeneous, then α_f is the constant function.

Proof. The distribution $\mathcal{Y}$ is invariantly defined being characterized by the property:

$$\mathcal{Y}_P = \{Y \in T_P M : R(Z_1, Z_2, Z_3, Y) = 0 \quad \text{for all} \quad Z_i \in T_P M\} \,.$$

The subspace $\mathcal{X}$ on the other hand is not invariantly defined. Denote the standard projection by π from $T_P M$ to $T_P M / \mathcal{Y}_P$. As

$$L(\cdot, \cdot) = 0, \quad R_f(\cdot, \cdot, \cdot, \cdot) = 0 \quad \text{and} \quad \nabla R_f(\cdot, \cdot, \cdot, \cdot; \cdot) = 0$$

if any entry belongs to Y, these tensors induce corresponding structures $\bar{L}_f$, $\bar{R}_f$, and $\mathfrak{R}_f$ on $T_P M / \mathcal{Y}_P$ so that

$$L_f = \pi^* \bar{L}_f, \quad R_f = \pi^* \bar{R}_f, \quad \text{and} \quad \nabla R_f = \pi^* \mathfrak{R}_f \,.$$

If $\mathcal{B}$ is an admissible basis, then we may define a quadratic form $\phi_{\mathcal{B}}$ on $T_P M / \mathcal{Y}_P$ by requiring that $\{\pi X_1, ..., \pi X_p\}$ is orthonormal with respect to this quadratic form. We then have $\bar{R}_f = R_{\phi_{\mathcal{B}}}$. By Lemma 3.1, $\phi = \phi_{\mathcal{B}}$ is independent of the particular basis chosen and is invariantly defined. This defines a positive definite inner product on $T_P M / \mathcal{Y}_P$ which we use to raise and lower indices and to contract tensors. The

invariant α is then given by $||\mathfrak{R}_f||_\phi^2$ and is invariantly defined. Since the structures involved are preserved by isometries, the Lemma now follows. What we have done, of course, is to prove that the second fundamental form is preserved by a local isometry of (M, g_f) in this setting. $\qquad\square$

Proof of Theorem 1.1. In light of Lemma 3.2, to complete the proof of Theorem 1.1, it suffices to construct f so that α_f is constant on no open subset of $\mathbb{R}^p$; the fact that such f are generic will then follow using standard arguments. Let $f_{;i} = \partial_i^x f$, $f_{;ij} := \partial_i^x \partial_j^x f$, and so forth. We use Lemma 2.1 to see:

$$R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x) = f_{;il} f_{;jk} - f_{;ik} f_{;jl},$$
$$\nabla R(\partial_i^x, \partial_j^x, \partial_k^x, \partial_l^x; \partial_n^x) = \partial_n^x \{f_{;il} f_{;jk} - f_{;ik} f_{;jl}\}.$$

Let $\Theta = \Theta(x_1)$ be a smooth function on $\mathbb{R}$ so that $|\Theta_{;11}| < 1$. Set

$$f(x) := \tfrac{1}{2}\{x_1^2 + ... + x_p^2\} + \Theta(x_1).$$

We may then compute, up to the usual $\mathbb{Z}_2$ symmetries, that the non-zero components of R_f and of ∇R_f are:

$$\begin{aligned}
R_f(\partial_1^x, \partial_i^x, \partial_i^x, \partial_1^x) &= 1 + \Theta_{;11} && \text{for} \quad 2 \le i \le p, \\
R_f(\partial_i^x, \partial_j^x, \partial_j^x, \partial_i^x) &= 1 && \text{for} \quad 2 \le i < j \le p, \\
\nabla R_f(\partial_1^x, \partial_i^x, \partial_i^x, \partial_1^x; \partial_1^x) &= \Theta_{;111} && \text{for} \quad 2 \le i \le p.
\end{aligned}$$

Consequently after taking into account to normalize the basis for the tangent bundle suitably, we have

$$\alpha_f = \tfrac{4(p-1)\Theta_{;111}^2}{(1+\Theta_{;11})^3}.$$

It is now clear that the metric g_f will not be locally homogeneous for generic Θ.
$\qquad\square$

Remark 3.3. Let $\{\vec{u}_1, ..., \vec{u}_p, \vec{v}_1, ..., \vec{v}_p\}$ be a basis for a vector space V of dimension $2p$. Define an innerproduct $(\cdot, \cdot)$ and an algebraic curvature tensor R on V whose non-zero entries are

$$(\vec{u}_i, \vec{v}_j) = \delta_{ij} \quad \text{and} \quad R(\vec{u}_i, \vec{u}_j, \vec{u}_k, \vec{u}_l) = \delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl}.$$

Then by Lemma 2.2, $(V, (\cdot, \cdot), R)$ is a model for the metric and curvature tensor of all the manifolds (M, g_f) considered above. If we set $\Theta = 0$, then

$$f_0 = \tfrac{1}{2}\{x_1^2 + ... + x_p^2\}.$$

Since $\nabla R = 0$, (M, g_{f_0}) is a symmetric space and hence locally homogeneous. This shows that $(V, (\cdot, \cdot), R)$ is the model for a symmetric space. Thus there exist pseudo-Riemannian manifolds which are not locally homogeneous whose metric and curvature tensor is modeled on those of a symmetric space.

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